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MEASUREMENT OF GROWTH AND EFFICIENCY IN ARITHMETIC (*Continued*)

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Taking up now the analysis of the tests, Table III gives the grade scores in the four operations. They have been recomputed from the actual results on a basis of uniform grade membership and equalized on the basis of equal thirteenth-grade attempts in each operation. The scores for fundamentals are by examples. The scores by points show no fundamental differences. The scores for subtraction, which occurs incidentally in division, have the same general relations as those for division and are not given. The scores in reasoning are by points. It is to be remembered, however, that a point in reasoning was one step of a problem, really a simple example by itself. The results are given graphically in Plots 6 and 7.

TABLE III

GRADE SCORES IN THE FOUR OPERATIONS

Recomputed on basis of uniform grade membership and equalized on basis of equal thirteenth-grade attempts in each operation.

GRADE	FUNDAMENTALS BY EXAMPLES						REASONING BY POINTS							
	Attempts			Rights			Attempts				Rights			
	A	M	D	A	M	D	A	S	M	D	A	S	M	D
3	44	4	0	6	0	0	60	52	77	81	43	37	43	62
4	50	35	12	34	22	5	99	90	105	128	97	77	87	108
5	52	46	43	30	36	23	213	209	172	234	164	157	117	181
6	64	48	40	40	34	31	228	221	220	250	159	120	141	156
7	62	55	50	44	42	40	259	209	201	242	175	167	161	206
8	70	71	67	48	54	50	234	248	225	252	196	210	192	234
9	64	60	58	53	44	47	238	251	236	264	205	232	180	238
10	74	69	70	54	58	46	240	253	239	267	213	217	202	250
11	66	63	56	50	54	46	256	273	263	274	232	251	240	257
12	68	64	60	49	53	47	242	240	239	260	215	220	224	238
13	74	74	74	58	61	64	273	273	273	273	250	237	231	256

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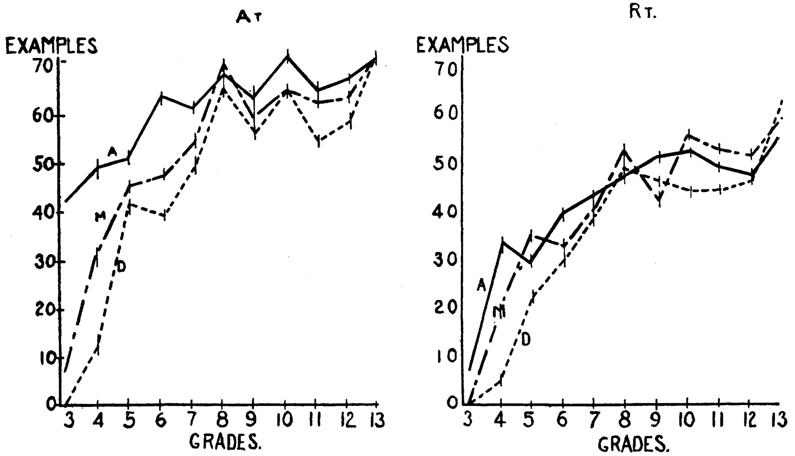
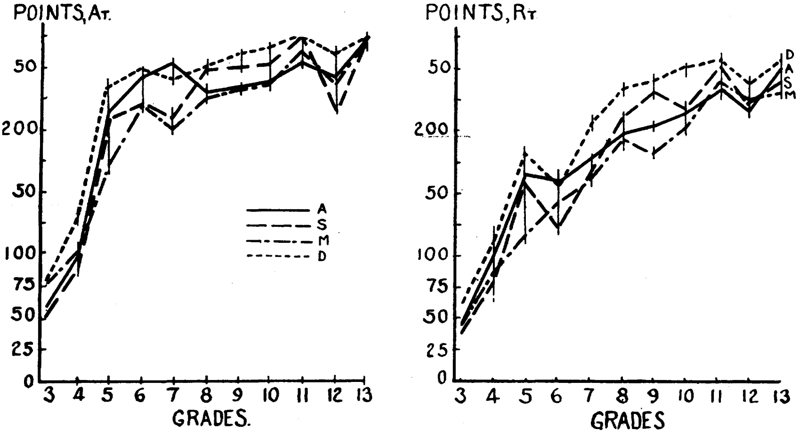
[1] For the first part of this paper see the *Elementary School Teacher* for October.

The curves show plainly the relative amount of development in each operation and the defects of the various grades. Thus, in Plot 6, fundamentals, in the fifth-grade drill has been



Plot 6.—Comparative scores in fundamentals by examples. Addition, multiplication, and division as shown in Table III.



Plot 7.—Comparative scores in reasoning by points. Four operations as shown in Table III.

given on the multiplication tables and long division, both learned in the grade before but mastered here, to the detriment of addition. Grade 9, on the other hand, is strong on addition and weak on multiplication and division. The form of the curve for any one operation would be very interesting if these were the results of one class for 13 years. Note the form of the curves in the high-school years. Algebra seems to help addition and multiplication which do not begin to decline before eleventh grade geometry, but does not aid division which declines from the eighth grade. However, the variation from grade to grade, which was partially eliminated in the combined results, is very evident here and prevents any such conclusions. For instance, the failure of the tenth grade in division was probably due to a grade defect and not to the lack of practice in algebra, for the thirteenth grade did better in division than anything else. The curves for reasoning permit similar diagnoses of defects and the remedy in each case is evident. Whether such analysis is of value can be told only after a year's remedial work and another test. The most evident fact to be inferred from these curves is that of grade variation and peculiarity. Because a class does well in addition it will not necessarily do well in any other operation, not even subtraction. The cause of such variation and peculiarities must be a subject for psychological investigation before much scientific teaching can be done. The curves obtained seem to the writer further evidence of the rudimentary and chaotic state of our present knowledge.

These results also have some bearing on the much-discussed question of the transfer of mental training from one subject to another. If skill in addition does not influence skill in subtraction, it would seem that there could be little hope of transfer of skill to any other subject. That something is transferred, however, we all know: what it is, is another story. These results seem to uphold the contention of certain psychologists against the value of special training in the development of general ability.

The results have, also, a bearing on the question of the relative degree of difficulty in the four operations. If the examples right are expressed as percentages of those attempted, we have the data presented in Table IV. These same facts are summarized in Table V.

These results make it probable that accuracy of work is a function of the development of the child, and in a way independent of his arithmetical training. Each grade seems to have a certain general degree of accuracy, which after the early years of learning does not seem to increase as one would expect. Within the limits of the accuracy of a class, all combinations of accuracies in the different operations are found, but the extremes

TABLE IV

Efficiencies in Four Operations in Fundamentals and Reasoning

Examples right expressed as percentages of examples attempted

Grade	Fundamentals			Reasoning			
	A	M	D	A	S	M	D
	per cent.	per cent.	per cent.	per cent.	per cent.	per cent.	per cent.
3	14	0	0	72	71	57	77
4	68	63	42	98	86	82	85
5	60	78	54	77	75	68	77
6	63	71	78	70	55	64	62
7	71	76	80	68	79	80	85
8	69	76	75	83	85	85	93
9	83	73	81	86	93	76	90
10	73	84	66	89	86	87	94
11	76	86	82	91	92	91	93
12	72	83	73	89	92	75	92
13	78	83	87	93	87	85	93

TABLE V

Relative Difficulty of Four Operations

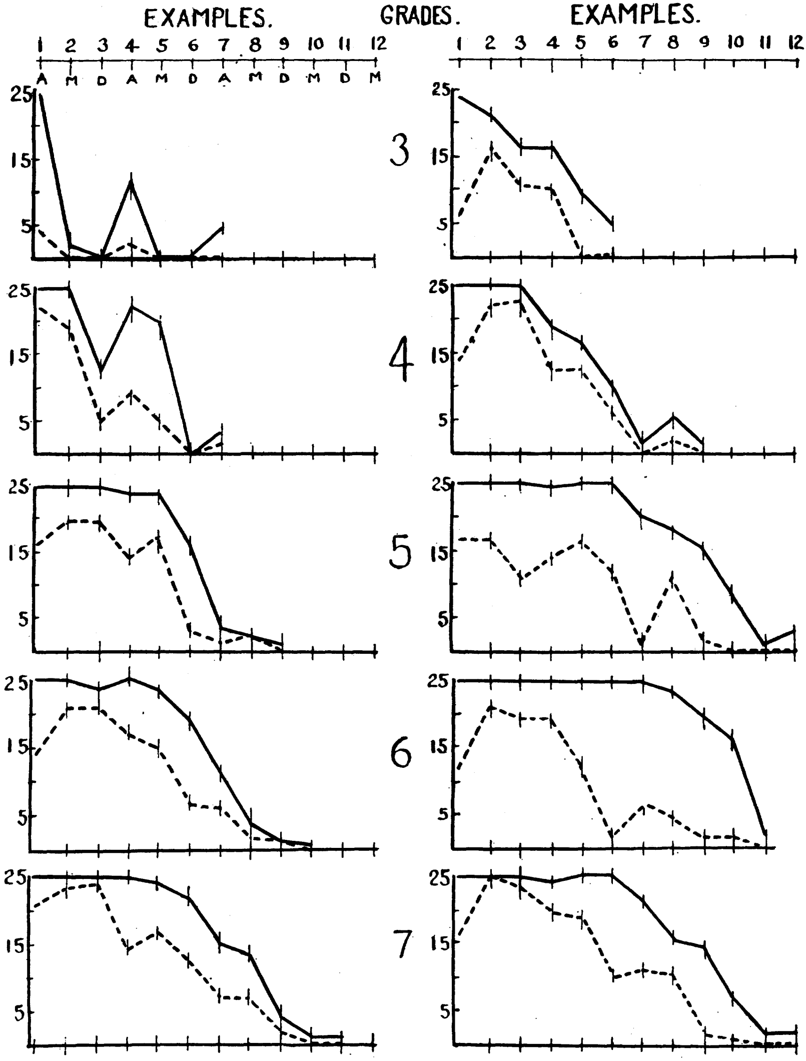
Derived from Table IV

Grade	Fundamentals											Totals			Conclusion
	3	4	5	6	7	8	9	10	11	12	13	A	M	D	
Easiest	A	A	M	D	D	M	A	M	M	M	D	3	5	3	M
Next		M	A	M	M	D	D	A	D	D	M	2	4	4	D, M
Hardest		D	D	A	A	A	M	D	A	A	A	6	1	3	A

Grade	Reasoning											Totals				Conclusion
	3	4	5	6	7	8	9	10	11	12	13	A	S	M	D	
Easiest	D	A	DA	A	D	D	S	D	D	DS	DA	4	2	0	8	D
Next	A	S	DA	M	M	MS	D	A	S	DS	DA	4	4	3	4	ASD
Next	S	D	S	D	S	MS	A	M	MA	A	S	3	5	3	2	S
Hardest	M	M	M	S	A	A	M	S	MA	M	M	2	2	7	0	M

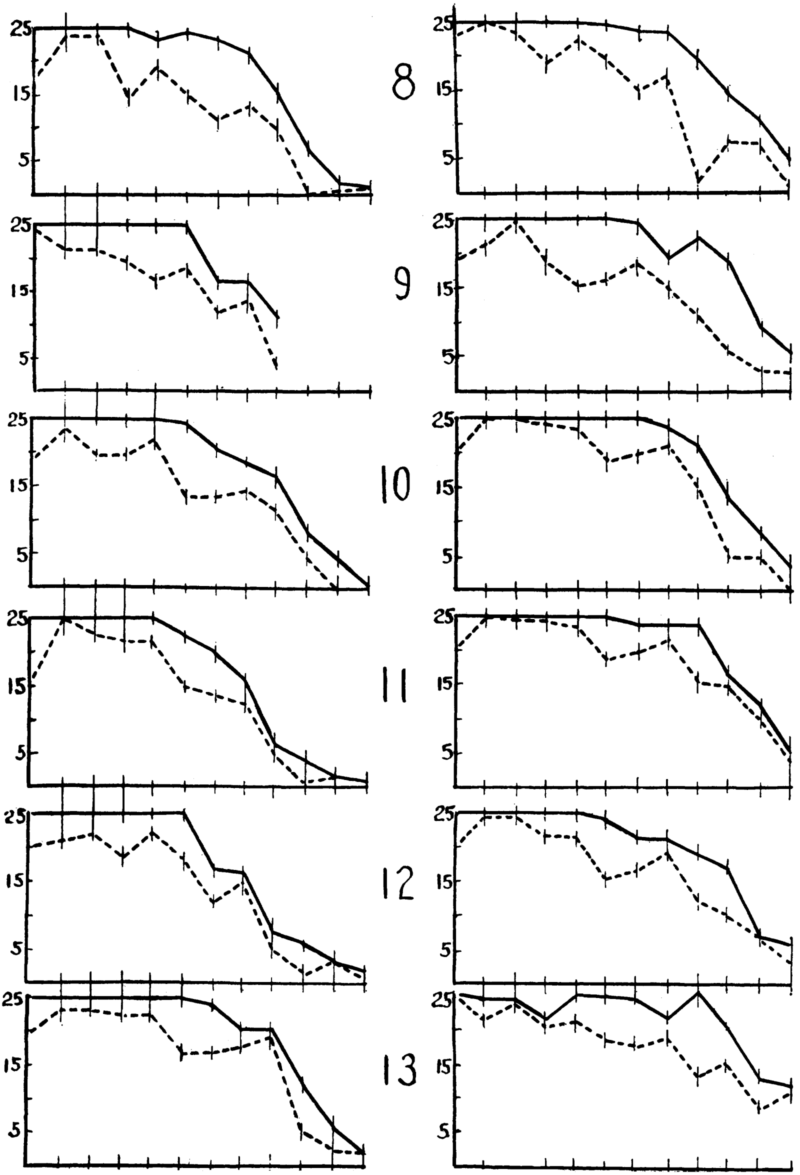
for each grade differ by about the same amount. So far as there is any indication of difference in difficulty between the different operations, the results agree with the conclusions of Dr. Stone: For fundamentals, addition being hardest, multiplication or division least difficult; for reasoning, multiplication hardest and division least difficult. In fundamentals, the reason is not hard to find. Much more has been made of the multiplication tables than of the addition tables. Multiplication and division are more complicated processes than addition and harder to teach. It often happens that addition is much neglected after the class is once fairly at work on long division. It seems practically certain that in the present state of our arithmetic teaching each operation and each part or division of a topic is learned by the child as a separate unrelated activity. There is no co-ordination, no welding of separate parts into one science of number, no appreciation of the meaning and purpose of arithmetic as a whole. Accordingly the incidental emphasis of the teacher on one topic or another, due to the varying mentalities of the different classes of the same grade leaves a lasting bias toward skill in one operation or another. It only remains to point out how, in later grade, a high-school class in algebra say, a weakness in one operation masked by a fair general ability, operates to make the best efforts of the teacher of no avail. What is apparently merely a lack of attention or care to detail of the algebra, is really due to a deep-rooted defect of previous training in some important particular skill. Both class and teacher attack the problem of correction blindly and at a place that may need no correction at all. The real benefit comes incidentally and all psychological experiments prove that the overcoming of unconscious habits through incidental unconscious training is very difficult and tedious. The value of complete, all-around training in the early grades in determining general ability in the later grades is very great. And whether or not development is complete and many sided can be told only by such tests and analyses as these.

Leaving now the question of ability, the analysis of the mistakes made will be taken up. The reason why a given ability was not obtained is a question of even greater pedagogical importance than that of what ability was attained. It is this question that the analysis should help answer.



PLOT 8.—Examples attempted and right in grades 3 to 7. See Table VI.

The first step is to gain a comprehensive view of the work actually done by each grade, and this is given in Table VI, plots 8 and 9, which show the examples attempted and right for each grade in both tests. The results have been corrected to a uniform grade membership of 25.



Plot 9.—Examples attempted and right in grades 8 to 13. See Table VI.

TABLE VI

Grade Records, Attempts and Rights, for Each Example in Both Tests

Results adjusted to uniform grade membership of 25

Part 1. Fundamentals

Examples

Grade	1		2		3		4		5		6		7		8		9		10		11		12	
	At.	Rt.	At.	Rt.	At.	Rt.	At.	Rt.	At.	Rt.	At.	Rt.	At.	Rt.	At.	Rt.	At.	Rt.	At.	Rt.	At.	Rt.	At.	Rt.
3....	25	4	2	0	0	0	12	2	0	0	0	0	4	0	0	0	0	0	0	0	0	0	0	0
4....	25	22	25	19	12	5	22	9	20	5	0	0	3	2										
5....	25	16	25	19	25	19	24	13	24	13	16	3	3	1	2	2	1	0						
6....	25	14	25	21	23	21	25	17	23	14	19	7	12	6	4	2	2	2	2	0				
7....	25	21	25	23	25	24	25	14	24	17	22	12	15	7	13	7	4	2	1	0	1	0		
8....	25	18	25	24	25	24	25	14	23	19	24	14	23	11	21	13	15	10	7	0	2	1	1	1
9....	25	24	25	22	25	22	25	19	25	17	24	18	17	11	17	13	10	4						
10....	25	19	25	23	25	19	25	19	25	22	24	13	20	13	18	14	16	12	8	5	4	0	1	0
11....	25	17	25	25	25	23	25	22	25	22	22	15	19	13	15	12	7	5	4	1	2	2	1	1
12....	25	20	25	21	25	22	25	18	25	22	25	18	17	11	16	14	8	5	6	2	3	3	2	1
13....	25	20	25	23	25	23	25	22	25	22	25	16	23	16	20	17	20	19	11	5	6	3	2	2

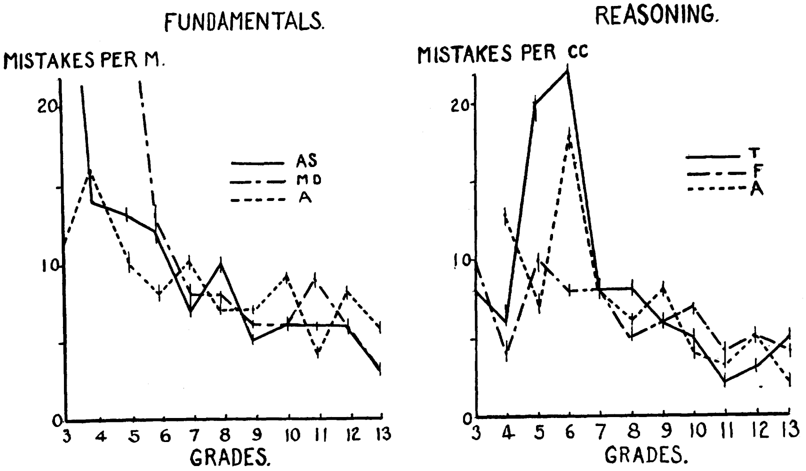
Part 2. Reasoning

Examples

3....	23	6	21	17	17	10	17	8	8	0	4	0												
4....	25	14	25	22	25	23	18	12	16	12	8	5	2	0	5	2	2	0						
5....	25	17	25	17	25	22	24	13	25	15	25	11	19	1	17	10	15	2	8	0	1	0	3	0
6....	25	12	25	21	25	19	25	19	25	12	25	2	25	6	23	4	19	2	15	2	2	0		
7....	25	17	25	25	25	23	24	19	25	18	25	10	21	11	15	10	14	2	8	1	2	0	2	0
8....	25	23	25	25	25	23	25	18	25	22	24	18	23	14	23	17	19	2	14	7	11	7	5	1
9....	25	19	25	22	25	25	25	18	25	15	25	16	24	18	19	15	22	11	18	6	8	3	6	3
10....	25	17	25	25	25	25	25	24	25	24	25	16	25	19	23	17	21	14	16	5	8	5	5	0
11....	25	21	25	25	25	24	25	24	25	23	25	18	23	19	23	21	23	15	17	14	13	10	6	4
12....	25	21	25	25	25	24	25	22	25	22	24	15	22	17	22	19	19	12	17	10	6	3	5	3
13....	25	24	24	22	24	24	22	20	25	21	25	18	24	17	22	18	25	13	20	15	13	8	12	11

The most striking fact shown by these results is again grade variation. The different forms of the fronts of the curves, from the convex forms of grades 6 and 7 to the concave variety shown in grade 12, fundamentals, signify different distributions of ability in the grades as will be shown later. Another striking fact is that for so few examples is the efficiency 100 per cent. Except where the example has been tried by but one or two, there is but one place in fundamentals (11th grade, 2d example) where all who tried a problem succeeded in getting it right. In reasoning the results are a little better, there being four cases of perfect scores for example 2 and two cases for example 3. These and the general forms of the curves make it apparent at once that the different examples varied greatly in their difficulty. For instance, in fundamentals, the curves for rights show a marked depression in many grades for examples 1, 4, and 7. But these are the addition examples and this fact but emphasizes the conclusion, reached before from a consideration of the scores for the various operations, that addition is the hardest operation. The excellence of grade 9 in addition could have been inferred from the fact that its curve shows no such depressions. So for other examples and grades.

The total mistakes made are given in Table VII. For comparison these are reduced to mistakes per thousand points. The results, however, give but another view of the growth in accuracy, previously shown in Plot 5; accordingly they are not plotted. Table VII also gives the analysis of mistakes. In fundamentals, addition and subtraction are grouped together (A and S), multiplication and division (M and D), and mistakes in copying or in carrying whether they occurred in one operation or another. These are called mistakes in attention (A). The purpose of this separation was to determine whether failure was due to a lack of knowledge of the addition and multiplication tables, or to a lack of ability to handle such knowledge in an example. The results have been reduced to those for uniform grade membership, and expressed as mistakes per thousand points for purposes of comparison. (Mistakes per two hundred points in reasoning.) It seemed better, also, to use individuals making mistakes rather than mistakes themselves, as a single individual, weak in her knowledge of the multiplication tables, was found to contribute many mistakes in multiplication in a single example. The scores have been called "mistakes" to save time and space, and have been much more accurately determined for fundamentals than for reasoning. In reasoning, as previously explained, mistakes were analyzed into mistakes of understanding (T), mistakes of fundamentals (F), and mistakes of attention (A).



PLOT 10.—Shows mistakes in each grade. For explanation of letters see foregoing text.

In both tests the total score in points is made the basis of the computations of mistakes per thousand in attention. While this is fair from one point of view, it masks the relative size of the different groups. Accordingly, the results have also been expressed in percentage of the total mistakes made for each grade. Plot 11 shows the same results graphically.

The most evident fact to be inferred from Plot 10 is the one noted previously, that after grade 7 the range of variation is fairly constant, and that within this range, the particular type of mistake made is a matter of grade peculiarity. The high values in the early grades are due, of course, to the fact that the learning process for the four operations has not been completed,

TABLE VII

FUNDAMENTALS

Grade							Per M			Percentage of TM		
	TM	MM	IM	AS	MD	A	AS	MD	A	AS	MD	A
3	81	89	43	33	0	10	36	0	11	78	0	21
4	76	48	58	16	17	25	14	50	16	28	39	43
5	97	36	73	22	26	25	13	34	10	30	36	34
6	98	32	61	25	12	24	12	13	8	41	20	39
7	69	21	55	15	8	32	7	8	10	27	15	58
8	85	20	70	28	11	31	10	8	7	40	16	44
9	53	15	43	10	7	26	4	6	7	23	16	61
10	68	15	65	17	8	40	6	6	9	26	12	62
11	49	14	42	15	10	17	6	9	4	36	24	40
12	66	17	53	15	7	31	6	6	8	28	13	59
13	57	12	45	12	4	31	3	3	6	27	5	69

REASONING

Grade						Per CC			Percentage of TM		
	TM	MM	MT	MF	MA	T	F	A	T	F	A
3	51	255	8	10	33	8	10	33	15	20	65
4	37	118	9	6	22	6	4	13	24	16	60
5	109	177	59	29	21	20	10	7	54	27	19
6	160	201	73	25	62	22	8	18	45	16	39
7	75	118	24	26	25	8	8	8	32	35	33
8	69	100	29	18	22	8	5	6	42	26	32
9	72	100	24	21	27	6	6	8	33	29	38
10	57	80	18	24	15	5	7	4	32	42	26
11	38	49	9	16	13	2	4	3	24	42	34
12	48	67	12	18	18	3	5	5	24	38	38
13	50	64	20	17	13	5	4	3	40	34	26

KEY

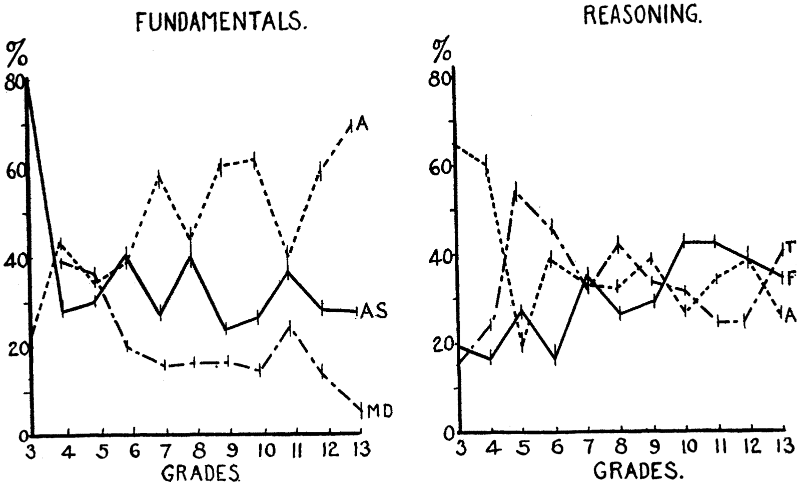
TM=Total Mistakes. MM=Mistakes per 1000. IM=Individuals making mistakes. AS=Individuals making mistakes in addition and subtraction. MD=Same, multiplication and division. A=Same, Attention. MT=Mistakes in understanding. MF=Mistakes in fundamentals.

SUPPLEMENT DATA

Grade	Total Points		Points A and S	Points M and D
	F	R		
3	917	202	906	0
4	1567	317	1327	340
5	2583	613	1773	809
6	3059	672	2136	923
7	3338	633	2317	1018
8	4324	690	2949	1376
9	3566	720	2455	1112
10	4426	720	3009	1416
11	3602	768	2510	1093
12	3903	712	2703	1200
13	4820	791	3263	1556

and the curves show plainly just where the learning of the different tables is done. In fundamentals, there is again not the evidence of growth that might be expected.

Plot 11 is more significant, as it indicates at once just where remedial work is to be done. In reading this plot however, care must be taken to consult the table, or Plot 10, to determine the actual amount of the mistakes in each group, in order not to be misled by the changes in the relative size of each group. Plot 11 shows that in fundamentals at grade 5 the size of the three groups is the same. That is, the child can direct her attention to her work just about as successfully as she can call upon a knowledge of the various tables needed in this test. Absolutely, the knowledge lags behind the attention as shown in Plot 10; for the test does not make equal demands upon all the operations. From grade 5 on, multiplication receives its proper training, so the curve declines until the mistakes in multiplication become a very small part of the whole. Addition improves so slowly that the percentage the addition mistakes are of the whole remains fairly constant. The child, however, receives, it would seem, no training in the use of her knowledge, and knows how to direct her attention no better at the higher grades than at grade 5. Conse-



PLOT 11.—For explanation see Table VII and key following table.

quently the relative percentage of the mistakes rises higher and higher. Even absolutely, in mistakes per thousand, the mistakes in attention outnumber the rest in five of the grades from the 7th to the 13th.

In reasoning the subject-matter is different and no direct comparison of the results can be made. It is interesting to note, however, that both in mistakes per cc and in percentages, in five of the grades mistakes in attention or mistakes in fundamentals which is due to the same cause are greater than mistakes in understanding.

This point, it seems to the writer, is an important one. Whatever the explanation, there can be no question of the fact that of all the individuals making mistakes at any given time in a class, at least one-third, and usually two-thirds, will be making mistakes in carrying or copying. The writer well knows that in discovering these mistakes there was much room for error. Yet it is to be remembered that the fact itself was made so evident in the general scoring that a second examination was made for the special purpose of separating these mistakes from the others. Also, the same method was used throughout, and should operate no more unjustly in one grade than in another. It is possible, and may be worth while, to carry the analysis farther.

The last subject to be considered in connection with the test, and to the writer the most important subject, is that of the distribution of ability in the grades. The total scores cover up many important facts and it is important, if any use is to be made of the results in determining future growth, that the present condition of each grade be exactly known.

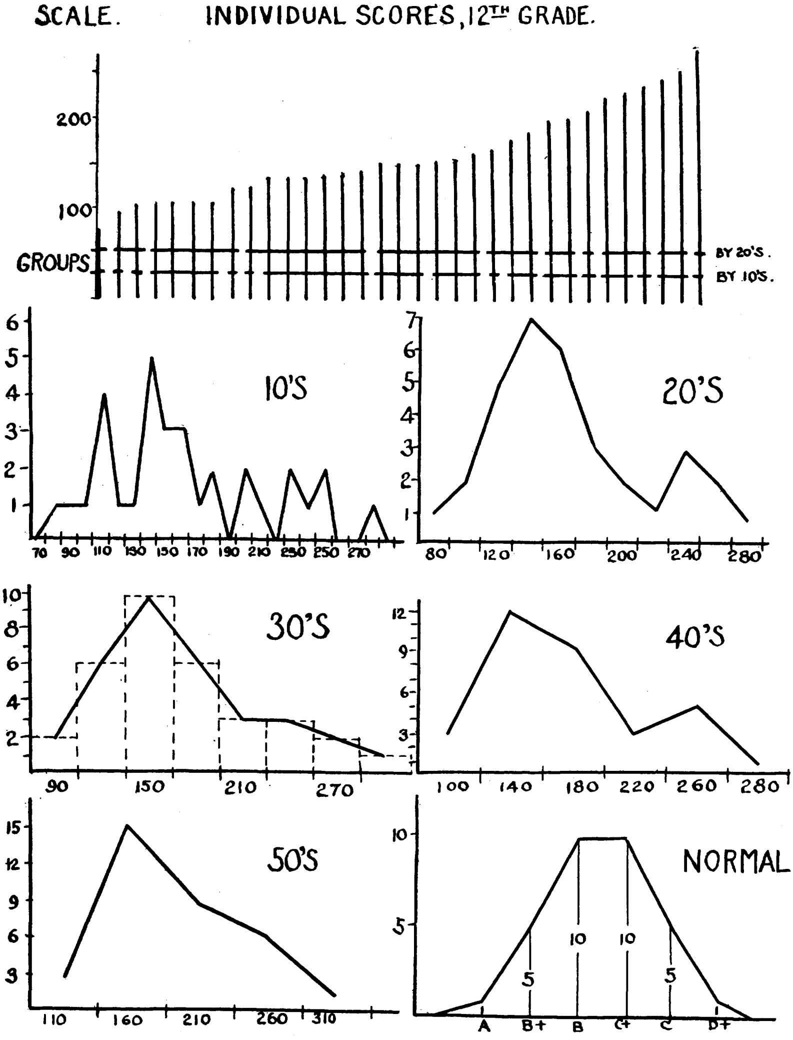
The twelfth and largest grade has been selected for a detailed study. In Table VIII and Graph 12 are given the individual scores of the 33 members of this grade grouped in various ways. At the top of Plot 12, the individual scores are represented by straight lines drawn to scale. It is evident at once that the extreme variation found in the different grades is also found in the individuals composing the grades. Some degree of variation was to be expected; the question is, how much is normal. The lowest score given is 71; the highest 274: yet both were "passed" through eight years of arithmetic training, to say nothing of the training of the high-school years in mathematics and science. Now a "pass" is supposed to stand for an efficiency of 70 per cent. at the very lowest, and the mark aimed at is 85–90 per cent. Either the system of marking and examining is very defective or there has been marked deterioration during high-school years in a large part of the class. Perhaps this variation can be made most plain by computing from the normal curve in Plot 4 the average value of the individual score in each grade

TABLE VIII

INDIVIDUAL SCORES AND GROUPINGS, GRADE 12. POINTS ATTEMPTED

INDIVIDUAL SCORES	By 10's		By 20's		By 30's		By 40's		By 50's	
	Score	Frequency	Score	Frequency	Score	Frequency	Score	Frequency	Score	Frequency
71.....	71– 80	1	61– 80	1	61– 90	2	61–100	3	50–100	3
88.....	81– 90	0	100	2	120	6	140	12	150	15
96.....	91–100	1	120	5	150	10	180	9	200	8
103.....	101–110	4	140	7	180	6	220	3	250	6
103.....	111–120	1	160	6	200	3	260	5	300	1
103.....	121–130	1	180	3	240	3	300	1		
103.....	131–140	6	200	2	270	2				
118.....	141–150	3	220	1	300	1				
122.....	151–160	3	240	3						
131.....	161–170	1	260	2						
131.....	171–180	2	280	1						
131.....	181–190	0								
135.....	191–200	2								
136.....	201–210	1								
139.....	211–220	0								
145.....	221–230	2								
145.....	231–240	1								
145.....	241–250	2								
151.....	251–260	0								
153.....	261–270	0								
158.....	271–280	1								
161.....										
171.....										
180.....										
194.....										
199.....										
204.....										
221.....										
223.....										
234.....										
241.....										
250.....										
274.....										

and expressing the composition of grade 12 in terms of members of the other grades. If this is done it will be found that the



Plot 12.—See Table VIII.

scores show grade 12 to be composed of 1 member of third-grade ability, 7 members of fourth-grade ability, 12 members of fifth-grade ability, 3 of sixth, 1 each of seventh-, twelfth-, and thirteenth grade abilities, and 7 members whose individual scores are above the average of the thirteenth grade. Over half the class is thus seen to be of less ability than the average ability of the sixth-grade class. This is evidently a subject that needs to be followed through with every grade. The method of doing this is that usually followed in discussions of distribution. The different individual scores are grouped together and the frequency with which any one score occurs can then be expressed by a single number. It is evident that if single scores are used the frequencies will be small, and the number of groups large. By dividing the whole score into equal parts, the number of groups can be decreased and the frequencies increased. The size of the measure to be used in any discussion depends upon the number of individual scores to be studied as well as upon the range of the variation. The highest score made by any individual in fundamentals was 278 points, the lowest 0. There are 11 grades. To allow for some overlapping, it seemed advisable to have more divisions of the score than there were grades, and yet a coarse measure was needed on account of the great variation. On the representation of the individual scores on Plot 12, the heavy black cross lines show the grouping when the total score is divided into periods of 10 points each, and into 20's. These groupings furnish the data for the distribution curves by 10's and 20's given immediately underneath. The divisions of the score are shown along the base line. One hundred and ten in the first graph meaning a group of all the scores from 101–110 inclusive. An inspection of the cross lines or of the table, will show four scores that fall within these limits. Accordingly the curve at this point rises to 4 above the base line. The size of the measure is thus shown along the base line and the frequency of any particular measure by heights above it.

An inspection of the distribution curve by 10's shows a succession of peaks, partially or wholly disconnected, while the same curve by 20's is a continuous curve. The normal distribution curve for any quality or ability in a large number of individuals has been determined, and for a group of 32 normal individuals would take the form shown at the bottom of Plot 12, marked normal. This means that in a class of 32 normal individuals, properly graded, there would be two groups of 10 each, to which the marks B and C+ could be given. On each side of these would be a smaller group of 5 each; one better, B+; one worse, C. Beyond these are two still smaller groups of one each, one of exceptional ability, A, the other very poor, D+. The real ability of the group ought to be taken as that of its central tendency, the central scores about which all the rest cluster, in this case the average of B and C+. In the

TABLE IX

DISTRIBUTION OF ABILITY

Based on individual scores in points. Each grade raised to fifty members

PART 1. FUNDAMENTALS

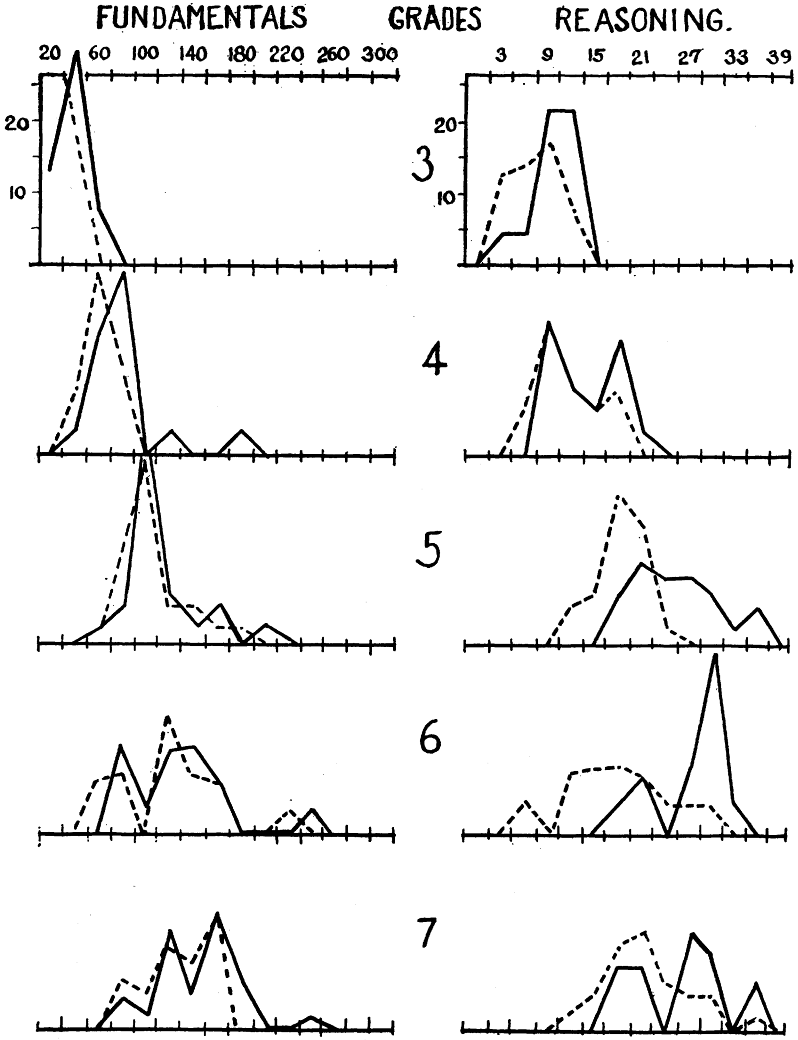
GRADE	SCORES AND FREQUENCIES													
	0–20	21-40	60	80	100	120	140	160	180	200	220	240	260	280
3 At.	13	29	8											
Rt.	36	20												
4 At.		3	16	25	0	3	0	0	3					
Rt.		9	25	12	3									
5 At.			2	5	28	7	2	5	0	2				
Rt.			2	11	25	5	5	2	2					
6 At.				12	3	11	12	7	0	0	0	3		
Rt.			7	8	0	16	8	7	0	0	3			
7 At.				5	2	14	5	16	7	0	0	2		
Rt.				7	5	11	9	16	0	0	0	2		
8 At.						5	7	5	19	2	7	2	2	
Rt.					2	5	7	7	14	1	7	2	2	
9 At.					3	14	3	14	8	3	6			
Rt.					3	17	3	14	6	6	3			
10 At.					2.	4	10	4	8	2	4	12	3	
Rt.					4	4	10	8	2	2	8	10	0	
11 At.				6	0	11	2	17	4	0	4	0	4	
Rt.				6	6	8	6	11	2	0	4	0	4	
12 At.				1	3	7	11	9	4	3	1	4	3	1
Rt.				3	6	6	9	12	3	4	4	0	0	1
13 At.						3	3	0	16	9	6	6	3	3
Rt.						3	3	6	13	6	6	3	3	

PART 2. REASONING

Grade	Scores and Frequencies											
	0–3	3–6	9	12	15	18	21	24	27	30	33	36
3 At.	4	4	21	21								
Rt.	12	13	17	8								
4 At.			19	9	6	16	3					
Rt.		6	19	9	6	9						
5 At.						7	11	9	9	7	2	5
Rt.				5	7	21	16	2				
6 At.						4	8	0	9	25	4	
Rt.		4	0	8	9	9	8	4				
7 At.						9	9	0	14	11	0	7
Rt.				2	5	12	14	7	5	5	0	2
8 At.					2	5	7	2	5	7	12	10
Rt.				2	2	5	10	7	10	2	7	5
9 At.							3	5	13	5	19	3
Rt.						5	13	5	8	16	3	
10 At.							11	0	9	14	9	7
Rt.							13	5	13	16	0	2
11 At.							2	2	10	10	14	12
Rt.							4	8	12	6	12	8
12 At.				2	0	2	5	3	2	18	2	12
Rt.			2	2	0	2	10	2	14	10	3	3
13 At.								7	3	13	8	18
Rt.						5	5	10	3	8	8	13

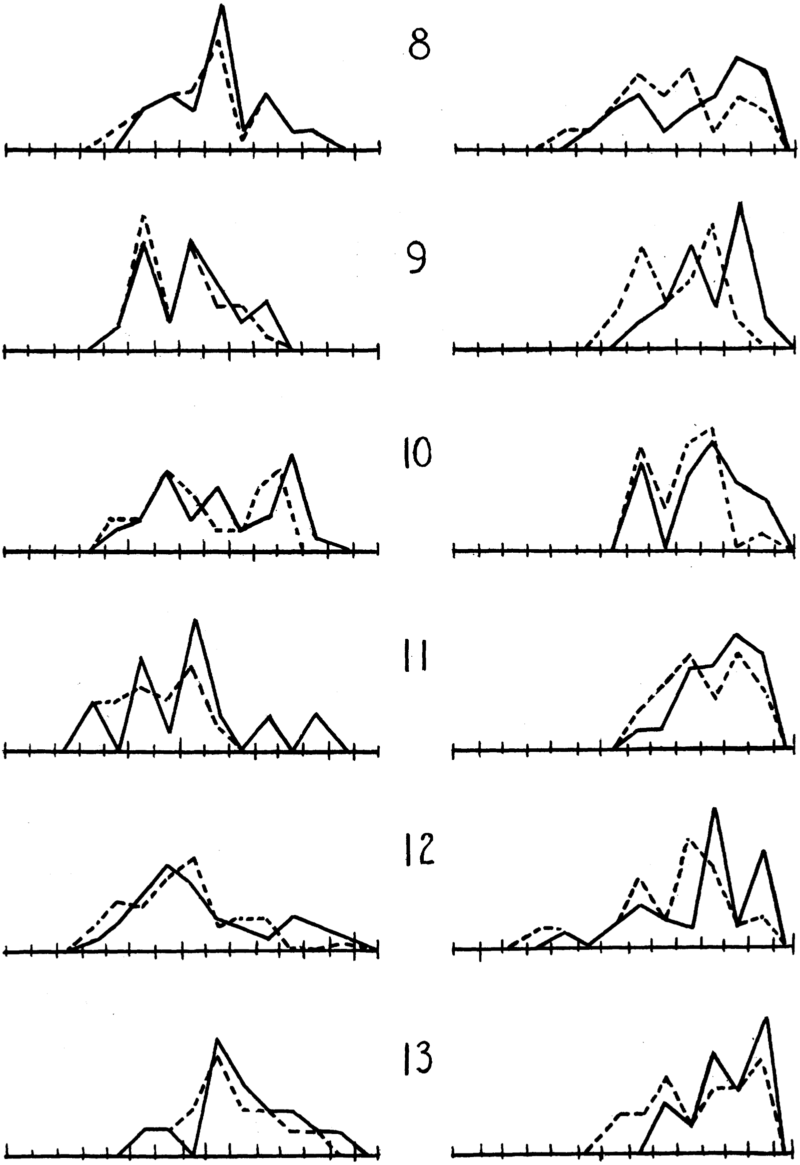
case of grade 12 above, it can be seen at once from the curve by 20's that the central tendency is from 120 to 140. Other curves are given to show the effect of coarser measures. In the case of the curve by 30's, representation by rectangles is also given as this method of drawing the distribution curve is sometimes used to indicate also the size of the measure. In all these curves, however, it will be noted that the individual abilities of grade 12 are in the main grouped about a low score but that there is in addition a smaller group of much greater ability.

For the general study of the distribution of ability in the grades, the measure is taken as 20 points for fundamentals and 3 points for reasoning. This gives 16 groups for fundamentals and 12 for reasoning. Each grade is raised to 50 members for convenience in plotting and all are plotted on the same scale.



PLOT 13.—Showing distribution of grades 3 to 7.

The full line represents attempts and the dotted line rights in every case. The curves for fundamentals and reasoning are given side by side in order that the distribution of the same grade in the two tests may be compared. The graphs will be known as Plots 13 and 14 and the data for it is given in Table IX.

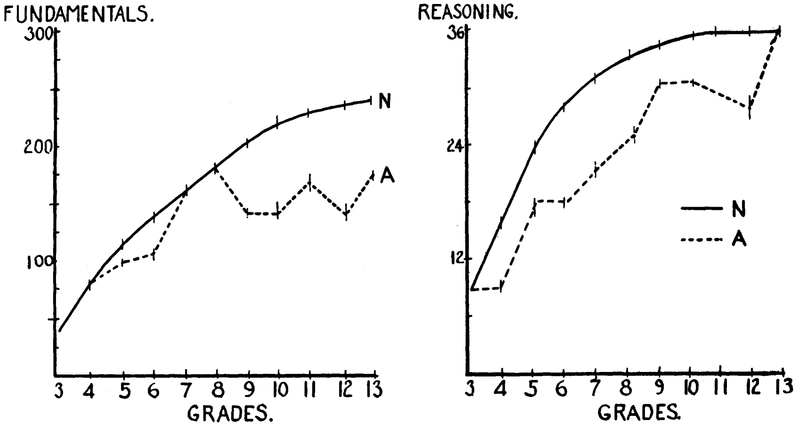


Plot 14.—Showing distribution of grades 8 to 13.

A very hasty glance at Plot 14 reveals a very irregular distribution of ability in all grades after the fifth, and it seems more than likely that the extent of the variation is much greater than would be normal under any kind of scientific study of individual abilities and needs. It seems to the writer that the determination of the causes of the wide variation in individuals, and the marking out of certain broad classification of individuals according to their peculiar abilities together with the determination of special training adapted for each class is the immediate problem of psychology upon whose solution any beginning of a scientific pedagogy must depend. For it is evident from the Plot that in the early grades the classes are units, but that as the pupils pass from grade to grade, individuals react to the training in different ways, so that the unity of the grade is broken up. It has long been evident that individuals in a class learn enough to pass examinations in very different ways. Some learn understandingly; some mechanically; that is, so that they can do the work set by a particular teacher in a certain way or place, but not so they can use the knowledge gained in any other place or for any other teacher. Whatever the explanation, a poor class, as grade 6, is here found to have a distribution showing several groups of very different abilities, while a good class, as grade 11, reasoning, is here shown to approach the normal distribution. This corresponds with teaching experience. Any teacher working with a small enough class to know the individuals, is conscious of the spur of a compact group of evenly graded children and the drag of the struggle with groups of individuals requiring special handling.

A careful study of the curves for fundamentals will show that the central tendency of the eighth grade is equaled by a large group in no grade except the thirteenth. A small group in the tenth grade is the only exception of any size. This seems to mean that the whole eighth grade has reached as high a degree of ability at the eighth grade as that attained by the high-school grades. The extent of the distribution in this class is greater than would seem right but otherwise it will make a good grade from which to determine what happens to ability in arithmetic during the high-school years.

The curves of Plot 15 permit a score being selected as the typical individual score for a grade; also, another score as the probable individual score that might be reached by the average of the class if all received the same benefit from their training as the best group. These values are given in Table X and their graph, 15, gives another possible normal and the departure of the present state of the school from it. It will be noted that the values in these curves are higher than those in the normals previously established and in most cases, the actual condition somewhat lower. These normals, the one of grade ability and the other of individual ability, furnish standards that make it worth while to attempt special remedial work with both individuals and grades. Their actual value can only be told by use.



PLOT 15.—Showing normal and actual for each grade. See Table X.

TABLE X

NORMAL AND ACTUAL INDIVIDUAL SCORES DERIVED FROM PLOT 15

	GRADES										
	3	4	5	6	7	8	9	10	11	12	13
Attempts—											
Fundamentals, normal	40	80	...	...	160	180	...	220	...	...	240
Fundamentals, central tendency	40	80	100	110	160	180	140	140	160	140	180
Rights—											
Reasoning, normal	9	15	24	...	...	33	...	...	...	...	36
Reasoning, central tendency	9	9	18	18	21	24	30	30	30	27	36

From the foregoing tables and plots, it will be seen that this test, while originally purely accidental and experimental in character, and not designed for the close analysis to which it has been subjected, has, nevertheless, yielded many results of value to the writer in his work of supervision. It has shown the need for greater knowledge of what is actually taking place in the child's mind as he passes through grade after grade. It has made plain the uncertainty of the product of the present educational system as well as the value of complete and many-sided training in the early grades if there is to be great ability in the later grades. More than all else it has proved, conclusively to the writer at least, that it is practicable to measure, not only the general condition of arithmetic teaching throughout a school, the growth in ability and efficiency from grade to grade, the defects and needs of any one grade or individual, but the *effects of changes in method or procedure* as well. By a series of tests, through a number of years, it ought to be possible to build up a real science of teaching and to determine by strict experimental methods the truth or falsity of any educational hypothesis. Nor is the method applicable to mathematics alone. In any subject taught in many successive grades, similar comparative testing will not only tend to produce increased efficiency but it will, judging by the results of the present test, furnish a connective thread of growth in the fundamentals of the subject that will produce a unity that is sadly lacking in all present pedagogical effort.